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TECHNICAL NOTE

Rayleigh surface waves in an idealised partially saturated soil

J. YANG*

KEYWORDS: dynamics; elasticity; partial saturation; stiffness; vibration

INTRODUCTION

The Rayleigh wave is a surface wave resulting from interference of the compressional and shear waves at the ground surface. In the vibration of foundations at the ground surface, a large proportion of the input energy is transmitted away by the Rayleigh wave, which decays much more slowly with distance than the compressional and shear waves (Richart *et al.*, 1970), suggesting that the Rayleigh wave is of primary concern in the design of machine foundations. Of particular interest is the application of Rayleigh waves in geotechnical site characterisation. Over the past three decades useful methods have been developed that allow the *in situ* stiffness of soil to be measured conveniently (Stokoe *et al.*, 1994). Modern surface wave techniques have evolved with the use of multi-channel receivers and more advanced signal-processing algorithms.

As documented in Stokoe *et al.* (1994) and Kramer (1996), theoretical developments and practical applications of Rayleigh waves in geotechnical engineering are based mainly on classical elasticity theory. That is, the soil is treated as an elastic solid continuum. This assumption may not well represent real cases where the soil is a multi-phase material. As a better approximation, soil is often modelled as a water-saturated poroelastic material, which can, according to Biot's theory (Biot, 1956), sustain three types of body wave: two compressional waves and one shear wave. Because of the complex interactions between the solid and fluid phases, the propagations of all three waves are frequency dependent and dissipative.

Following Biot's pioneering work, the problem of Rayleigh waves in a poroelastic saturated half-space received considerable attention (e.g. Deresiewicz, 1962; Mei & Foda, 1981; Tajuddin, 1984), mainly in the fields of geophysics and acoustics, where rock-like or artificial porous materials are of primary concern. Most of these works ignored the viscous coupling between the solid and fluid phases: in other words, the permeability of the porous material concerned was assumed as being infinity. Apparently, this assumption is not sufficiently reasonable. Moreover, it is noted that some results from prior studies are even conflicting and hence confusing. For example, the work by Tajuddin (1984) implies that the phase velocity of Rayleigh waves can be increased by a factor of 1·5 to 4 in saturated porous media compared with single-phase elastic solids, raising a question over the applicability of classical elasticity.

For the above reason, Yang (2001) revisited the Rayleigh wave problem with particular attention to the Rayleigh wave velocity in common soils such as sand, gravel and clay. The formulation of the analysis accounts for both viscous and inertia couplings between the solid and fluid parts. It has been shown that, in the low-frequency range, the Rayleigh wave velocity in a half-space is independent of frequency, and it is indeed—as predicted by classical elasticity—slightly lower than the shear wave velocity.

In many situations, near-surface soils are not fully saturated because of fluctuating groundwater levels associated with natural or man-made processes. The condition of partial saturation may have a significant impact on seismic site response as well as on soil liquefaction. For example, an unsaturated surface layer can cause much greater amplification in vertical ground motion (or approximately compressional wave motion) than a saturated soil layer, as indicated by the study of the three-dimensional downhole array records from the 1995 earthquake in Kobe, Japan (Yang & Sato, 2000). An analysis of the response of a semi-infinite soil medium to incident shear waves shows that the saturation state of the soil may have a large influence on the surface displacements in both horizontal and vertical components (Yang, 2002).

In view of these recent findings, and the fact that Rayleigh waves are the result of interfering compressional and shear waves, a great concern naturally arises over the potential influence of the saturation state of subsoil on the propagation of Rayleigh waves. The issues of primary concern include:

(*a*) whether the Rayleigh wave velocity is affected by the saturation state
(*b*) what the influence of saturation is on the displacements of Rayleigh waves
(*c*) how the soil particle trajectory is related to the saturation state.

In this concise paper, a first attempt is made to address these issues for the fundamental half-space model.

FORMULATION

The analysis is based on Biot's theory, which models the interactions between the soil skeleton and the pore fluid using the macroscopic laws of mechanics. In this regard, the wavelength of the involved wave should be large enough compared with the pore size of the soil. For most problems in soil dynamics, the frequencies of interest are not high, generally being less than 150 Hz. Therefore a typical shear wave velocity of 150 m/s will give rise to a wavelength of 1 m. This length should be adequately larger than the maximum pore size of most soils, and the continuum mechanics representation is applicable.

To account for the effect of partial saturation, the concept is adopted of homogeneous pore fluid, which assumes that the air is in the form of dispersed bubbles in pore water, and that the mixture of air and water can be approximately treated as an equivalent homogeneous fluid completely filling the voids with a pore pressure. As a result, the compressibility of the pore fluid can be related to the degree of saturation, the compressibility of pore water and the absolute

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* Department of Civil Engineering, The University of Hong Kong.

fluid pressure (Yang & Sato, 2000). Experimental and field case studies have shown that this treatment of partial saturation is simple yet effective in dealing with problems of wave propagation (Yang & Sato, 2000; Ishihara *et al.*, 2001). Nevertheless, it should be noted that in real partially saturated soil the interactions between different phases (e.g. matrix and osmotic suction) are complicated. Some constitutive models have been developed in the field of unsaturated soil mechanics to take account of these interactions (e.g. Thomas, 1987; Sills *et al.*, 1991; Fredlund & Rahardjo, 1993). A discussion of these models and their effectiveness in wave motion analysis is beyond the scope of the present paper but deserves to be seriously considered in future study.

Rayleigh waves are plane waves that can be thought of as combinations of the compressional and shear waves at the free surface of a half-space. They can induce both horizontal and vertical displacements. Referring to Fig. 1, the displacements of the solid phase in the x- and z-directions, denoted respectively by u_x and u_z, can be written in terms of two potential functions Φ_s and Ψ_s as

$$u_x = \frac{\partial \Phi_s}{\partial x} - \frac{\partial \Psi_s}{\partial z}, \quad u_z = \frac{\partial \Phi_s}{\partial z} + \frac{\partial \Psi_s}{\partial x} \tag{1}$$

The displacements of the pore fluid relative to the solid phase, denoted by w_x and w_z, can be expressed in a similar manner as

$$w_x = \frac{\partial \Phi_f}{\partial x} - \frac{\partial \Psi_f}{\partial z}, \quad w_z = \frac{\partial \Phi_f}{\partial z} + \frac{\partial \Psi_f}{\partial x} \tag{2}$$

where Φ_s and Ψ_s are potential functions for the solid part and Φ_f and Ψ_f are potentials for the fluid part.

The two patterns of motion can then be described in terms of potentials as follows.

Compressional motion:

$$(\lambda + \alpha^2 M + 2G)\nabla^2 \Phi_s + \alpha M \nabla^2 \Phi_f = \rho \frac{\partial^2 \Phi_s}{\partial t^2} + \rho_f \frac{\partial^2 \Phi_f}{\partial t^2} \tag{3}$$

$$\alpha M \nabla^2 \Phi_s + M \nabla^2 \Phi_f = \rho_f \frac{\partial^2 \Phi_s}{\partial t^2} + \frac{\rho_f}{n}\frac{\partial^2 \Phi_f}{\partial t^2} + \frac{\eta}{k'}\frac{\partial \Phi_f}{\partial t} \tag{4}$$

Distortional motion:

$$G\nabla^2 \Psi_s = \rho \frac{\partial^2 \Psi_s}{\partial t^2} - \rho_f \frac{\partial^2 \Psi_f}{\partial t^2} \tag{5}$$

$$\rho_f \frac{\partial^2 \Psi_s}{\partial t^2} + \frac{\rho_f}{n}\frac{\partial^2 \Psi_f}{\partial t^2} + \frac{\eta}{k'}\frac{\partial \Psi_f}{\partial t} = 0 \tag{6}$$

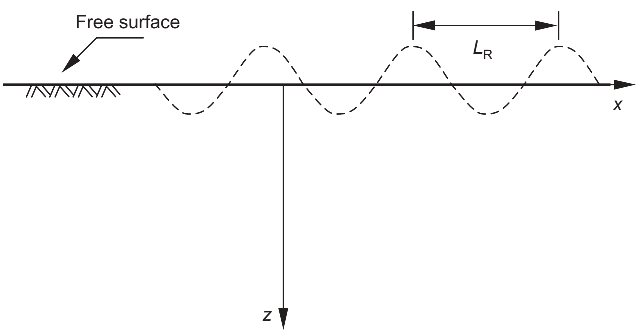

Fig. 1. Rayleigh waves in a half-space

where $\rho = (1 - n)\rho_s + n\rho_f$ is total mass density; ρ_s is the density of solid grains and ρ_f the density of pore fluid; n is soil porosity; G is the shear modulus and λ is the Lame constant of the solid frame; α and M are two parameters accounting for the compressibility of solid grains and pore fluid (Yang, 2002); η is fluid viscosity; and k' is permeability. Note that the dimension of k' is different from that of the permeability coefficient defined in soil mechanics.

Assuming the wave is harmonic, and travelling in the x-direction with frequency ω and wave number p, the four potential functions can be expressed as

$$\Phi_s = \left(A_1 e^{-iq_1 z} + A_2 e^{-iq_2 z}\right)e^{i(\omega t - px)} \tag{7}$$

$$\Phi_f = \left(\delta_1 A_1 e^{-iq_1 z} + \delta_2 A_2 e^{-iq_2 z}\right)e^{i(\omega t - px)} \tag{8}$$

$$\Psi_s = B e^{-iq_3 z} e^{i(\omega t - px)} \tag{9}$$

$$\Psi_f = \delta_3 B e^{-iq_3 z} e^{i(\omega t - px)} \tag{10}$$

in which $i = \sqrt{-1}$; A_1, A_2 and B are amplitudes of potential functions; δ_1, δ_2 and δ_3 represent the ratios between the amplitudes of the potentials for solid and those for fluid (Yang, 2002); and q_1, q_2 and q_3 can be expressed as

$$q_i^2 = l_i^2 - p^2 (i = 1, 2, 3) \tag{11}$$

where l_1, l_2 and l_3 are complex quantities relating to the velocity of the first compressional wave, V_1, the velocity of the second compressional wave, V_2, and the shear wave velocity, V_3, by

$$\frac{1}{V_i} = \mathrm{Re}\left(\frac{l_i}{\omega}\right)(i = 1, 2, 3) \tag{12}$$

in which Re( ) signifies the real part of the complex quantity.

The boundary conditions for the existence of Rayleigh waves specify that the surface is free of stress and free draining: that is, $\sigma_z = \tau_{xz} = p_f = 0$ at $z = 0$, where σ_z and τ_{xz} are normal and shear stresses respectively, and p_f is pore pressure. With constitutive relations, the boundary conditions can be further given in terms of displacements as

$$(\lambda + \alpha^2 M)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_z}{\partial z} - \alpha M\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z}\right) = 0 \tag{13}$$

$$\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \tag{14}$$

$$\alpha M\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - M\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z}\right) = 0 \tag{15}$$

By enforcing the above conditions, the secular equation for the Rayleigh wave in a homogeneous partially saturated soil can be derived as

$$\begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = 0 \tag{16}$$

The coefficients T_{ij} are given in the Appendix. From the secular equation the phase velocity of the Rayleigh wave can be calculated, and subsequently the displacements and particle motion can be determined.

RAYLEIGH WAVE VELOCITY

A significant application of Rayleigh waves in geotechnical engineering practice is to determine the stiffness of *in situ* soil. The application is based on the understanding that the Rayleigh wave velocity is approximately equal to the shear wave velocity, established from classical elasticity that models the soil as a solid continuum. Whether the Rayleigh wave velocity is affected by the saturation state of soil is an interesting issue that needs to be addressed.

Using the properties of a typical sand (Table 1), the phase velocity of the Rayleigh wave is computed as a function of the degree of saturation. For purposes of comparison, the values of the first compressional wave and shear wave velocity are shown in the same graph. It should be mentioned that, in the range of frequencies involved in soil dynamics, the propagation of both the first compressional wave and the shear wave is independent of frequency while the second compressional wave is highly attenuated, with its velocity approaching zero.

Figure 2 indicates that the influence of saturation is negligible for the shear wave velocity but very significant for the compressional wave velocity. As widely recognised, the latter may drop dramatically even for a slight decrease of full saturation. Of interest is the dependence of the Rayleigh wave velocity on the degree of saturation. Compared with the compressional and shear waves, the effect of saturation on the Rayleigh wave velocity appears to be intermediate. This result is reasonable, as the Rayleigh wave is the combination of the two waves. In general, the Rayleigh wave velocity decreases with decreasing degree of saturation, but the amount of the change is small.

RAYLEIGH WAVE DISPLACEMENT

The propagation of Rayleigh waves can produce both vertical and horizontal displacements in a soil medium. With the formulation derived, the displacement amplitudes in both components and at any specific depth can be determined in relation to the degree of saturation S_r, as shown in Fig. 3. Here the horizontal and vertical displacement amplitudes at

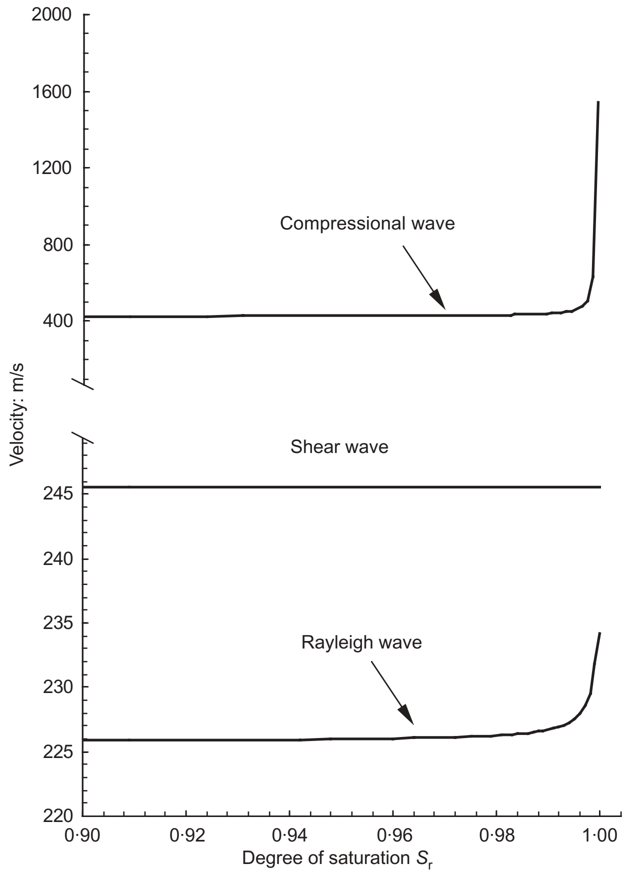

Fig. 2. Influence of saturation on wave velocity

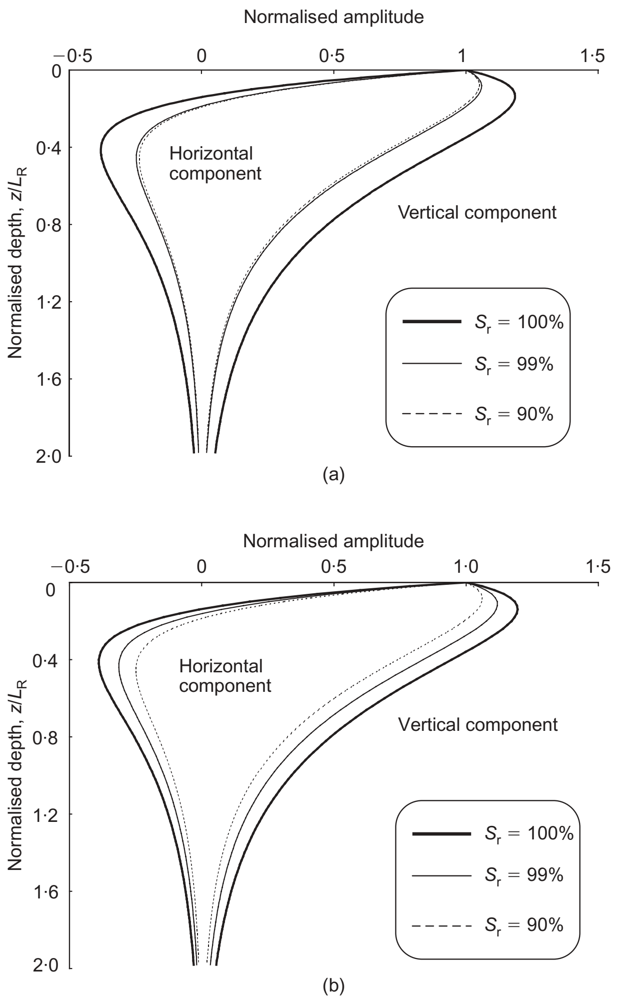

Fig. 3. Normalised amplitude against depth under different saturation conditions: (a) stiff soil, G = 120 MPa; (b) soft soil, G = 10 Mpa

Table 1. Properties of the sand used in analysis

Specific gravity	Porosity	Coefficient of permeability: m/s	Fluid viscosity: Pa s	Bulk modulus of pore water: MPa	Shear modulus: MPa	Poisson's ratio
2·65	0·40	10^{-5}	10^{-3}	2200	120	0·25

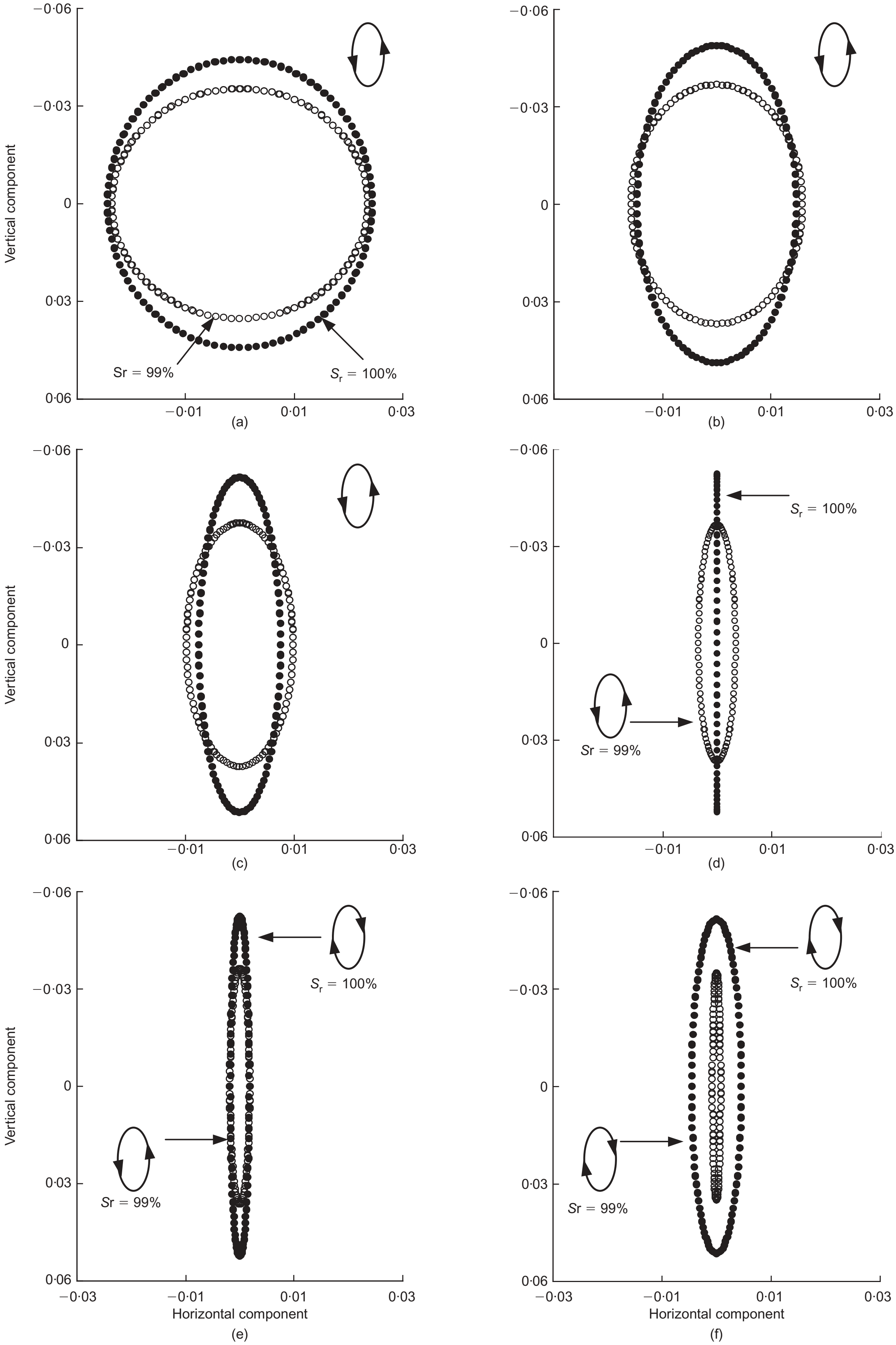

Fig. 4. Influence of saturation on particle trajectory: (a) $z = 0$; (b) $z = 0{\cdot}04L_R$; (c) $z = 0{\cdot}08L_R$; (d) $z = 0{\cdot}14L_R$; (e) $z = 0{\cdot}16L_R$; (f) $z = 0{\cdot}2L_R$

depths are normalised by the corresponding values at the ground surface, and the depth is normalised by the wavelength of the Rayleigh wave, L_R. In order to investigate simultaneously the influence of soil stiffness, a 'soft soil' case, where the shear modulus is assumed as 10 MPa with the remaining properties unchanged, is included in parallel.

It is evident from Fig. 3 that both the vertical and horizontal displacements approach zero at a depth of approximately twice the wavelength, implying the localisation of the wave motion in a thin layer near the surface. An impression of Fig. 3 is the difference between the cases of full and partial saturation: the Rayleigh wave can produce greater displacement amplitudes in a fully saturated soil than in a partially saturated soil. On the other hand, one may note that the difference between the two partially saturated cases (i.e. 99% and 90%) is relatively small, and the amount of difference seems to depend upon soil stiffness. For stiff soil ($G = 120$ MPa, or the corresponding shear wave velocity is about 246 m/s), the difference is negligible; but when the soil is soft ($G = 10$ MPa or the shear wave velocity is 71 m/s), the difference becomes appreciable.

RAYLEIGH WAVE TRAJECTORY

It is noted in Fig. 3 that the horizontal displacement will become zero when the vertical displacement reaches its maximum, implying that the horizontal and vertical displacements are out of phase by 90°. This can be observed more clearly in Fig. 4, where a series of loci of the particle motion generated by the Rayleigh wave on and near the surface at the position of $x = 0$ are illustrated for the stiffer soil. Two cases of saturation (i.e. 100% and 99%) are shown together for ease of comparison.

Clearly, in either the fully or partially saturated case, the path of the particle motion describes an ellipse. For the coordinate shown in Fig. 1 the motion is counterclockwise (retrograde) on or near the ground surface. At a depth of approximately $0{\cdot}2L_R$ the direction of rotation reverses: that is, the particle motion is clockwise (prograde). In the transition the motion will be purely vertical at the depth where the horizontal component vanishes.

An interesting feature of Fig. 4 is that even a slight decrease of full saturation may cause a marked change of particle trajectory, and the amount of change is related to the relative depth. At the surface, or at very shallow depths, the decrease of full saturation appears to produce a more significant influence on the vertical than on the horizontal component. As a consequence, the shape of the locus will be changed such that the major axis of the ellipse becomes shorter while the length of the minor axis remains almost unchanged.

It is of interest to note that at a depth of $0{\cdot}16L_R$ the particle motion under fully saturated conditions has already become *clockwise* but it remains *counter-clockwise* for partially saturated conditions. It is also found that, for depths approximately greater than $0{\cdot}2L_R$, the conditions of partial saturation cause a change in the size of the particle locus only, without appreciable change in the shape of locus. Because of space limitations, the results are not shown here.

CONCLUDING REMARKS

A first attempt has been made to identify the potential influence of the saturation state of subsoil on the propagation of Rayleigh surface waves. Analysis has been performed for an idealised model to address several fundamental issues. The results can be summarised as follows.

(*a*) The Rayleigh wave velocity decreases with decreasing saturation, but the amount of the change is very small. For a typical sand, the ratio of the Rayleigh wave velocity to the shear wave velocity at fully saturated conditions is approximately 0·95, and it reduces to a value of about 0·92 at a saturation of 90%.

(*b*) The influence of saturation appears to be more significant for particle displacement and trajectory. The Rayleigh wave can produce greater displacement amplitudes in a fully saturated soil than in a partially saturated soil, and the effect of saturation appears to be more significant for the vertical than the horizontal component.

(*c*) Even a slight decrease of full saturation may cause a marked change in the locus of the near-surface particle motion, including its size, shape and even the direction of rotation, implying the potential for using Rayleigh waves to assess the saturation conditions of *in situ* soils.

ACKNOWLEDGEMENTS

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APPENDIX

The coefficients in equation (16) are as follows:

$$T_{11} = \left(\frac{\lambda + \alpha^2 M}{2G} + \frac{\alpha M}{2G}\delta_1\right) l_1^2 + \left(l_1^2 - p^2\right)$$

$$T_{12} = \left(\frac{\lambda + \alpha^2 M}{2G} + \frac{\alpha M}{2G}\delta_2\right) l_2^2 + \left(l_2^2 - p^2\right)$$

$$T_{13} = p\left(l_3^2 - p^2\right)^{\frac{1}{2}}$$

$$T_{21} = 2p\left(l_1^2 - p^2\right)^{\frac{1}{2}}$$

$$T_{22} = 2p\left(l_2^2 - p^2\right)^{\frac{1}{2}}$$

$$T_{23} = 2p^2 - l_3^2$$

$$T_{31} = l_1^2(\delta_1 + \alpha)$$

$$T_{32} = l_2^2(\delta_2 + \alpha)$$

$$T_{33} = 0$$

NOTATION

G	shear modulus of solid frame
k'	permeability (unit: length2)
n	porosity
S_r	degree of saturation
u_x, u_z	displacements of solid part in the x-z plane
w_x, w_z	displacements of fluid part relative to solid part in the x-z plane
α, M	parameters accounting for the compressibility of constituents
δ_1, δ_2, δ_3	amplitude ratios between potentials for solid part and for fluid part
η	fluid viscosity
λ	Lamé constant of solid frame
Φ_f, Ψ_f	potential functions for fluid part
Φ_s, Ψ_s	potential functions for solid part
ρ	total density
ρ_f	density of fluid
ρ_s	density of solid grains
ω	angular frequency

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