State-Dependent Strength of Sands from the Perspective of Unified Modeling

J. Yang, M.ASCE,[1] and X. S. Li, M.ASCE[2]

Abstract: This paper discusses the state-dependent strength of sands from the perspective of unified modeling in triaxial stress space. The modeling accounts for the dependence of dilatancy on the material internal state during the deformation history and thus has the capability of describing the behavior of a sand with different densities and stress levels in a unified way. Analyses are made for the Toyoura sand whose behavior has been well documented by laboratory tests and meanwhile comparisons with experimental observations on other sands are presented. It is shown that the influence of density and stress level on the strength of sands can be combined through the state-dependent dilatancy such that both the peak friction angle and maximum dilation angle are well correlated with a so-called state parameter. A unique, linear relationship is suggested between the peak friction angle and the maximum dilation angle for a wide range of densities and stress levels. The relationship, which is found to be in good agreement with recent experimental findings on a different sand, implies that the excess angle of shearing due to dilatancy in triaxial conditions is less than 40% of that in plane strain conditions. A careful identification of the deficiency of the classical Rowe's and Cam-clay's stress–dilatancy relations reveals that the unique relationship between the stress ratio and dilatancy assumed in both relations does not exist and thereby obstructs unified modeling of the sand behavior over a full range of densities and stress levels.

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Introduction

It has been consistently observed in triaxial tests that, subjected to a shear under drained conditions, dense sand dilates accompanied by strain softening and loose sand contracts accompanied by strain hardening, as shown in Fig. 1. Whether a sand is in a loose or dense state depends not only on its density but also on the confining pressure applied. Moreover, for a sand that initially is either in the loose or dense state there exists an ultimate state of shear failure at which the volumetric strain rate is zero. This ultimate state, widely known as the critical state (Roscoe et al. 1958; Schofield and Wroth 1968), is characterized by a unique combination of critical void ratio and stress ratio of deviator stress to mean effective stress in a triaxial setting.

The density and pressure dependence of shear strength is one typical and important feature of sand behavior that needs to be taken into account in engineering design. Laboratory investigations have been made extensively into the combined influence of density and pressure on shear strength of sands (e.g., Comforth 1964, 1973; Lee and Seed 1967; Bishop 1971; Stroud 1971; Been and Jefferies 1985; Vaid and Sasitharan 1992; Verdugo and Ishihara 1996). Bolton (1986) presented a comprehensive review of the experimental data and suggested an empirical relation for estimating the peak angle of shearing resistance

$$(\phi_p' - \phi_{cs}') = 3[I_D(10 - \ln p') - 1] \tag{1}$$

where ϕ_p' = friction angle corresponding to the peak strength; ϕ_{cs}' = critical state angle of friction for constant volume shearing; I_D = initial relative density; and p' = mean effective stress measured in kilopascals at failure.

If it is intended only to compile results from a series of laboratory tests such as triaxial compression or plane-strain shear tests, it may not matter whether the peak friction angle is correlated with initial state rather than the state at which peak failure is reached. From the point of view of constitutive modeling (Wood 1990), however, the soil's perception of peak strength that it is capable of mobilizing should be updated according to its initial state and the failure stress cannot be regarded as a constant, input soil parameter. Furthermore, since there can be no restriction on the stress paths to which the soil may be subjected in modeling, changes in the void ratio and mean effective stress may well be involved. Therefore, a further investigation into the state-dependent strength of sands from the perspective of constitutive modeling shall be helpful and insightful, which is precisely the motivation of this work.

Rationally modeling the behavior of granular soils has been a challenging work. One of the fundamental issues in modeling the stress–strain–strength behavior of sands is the coupling between shear and volumetric strains, which can appropriately be described by the dilatancy, d, the ratio of plastic volumetric strain increment to plastic deviator strain increment. Based on the theory of least rate of internal work, Rowe (1962) showed that the dilatancy could be expressed as a function of stress ratio and the true angle of friction between the mineral surfaces of the par-

[1]Assistant Professor, Dept. of Civil Engineering, The Univ. of Hong Kong, Pokfulam Rd., Hong Kong, China; formerly, Technical Univ. of Berlin, Berlin, Germany.

[2]Associate Professor, Dept. of Civil Engineering, The Hong Kong Univ. of Science and Technology, Clear Water Bay, Hong Kong, China.

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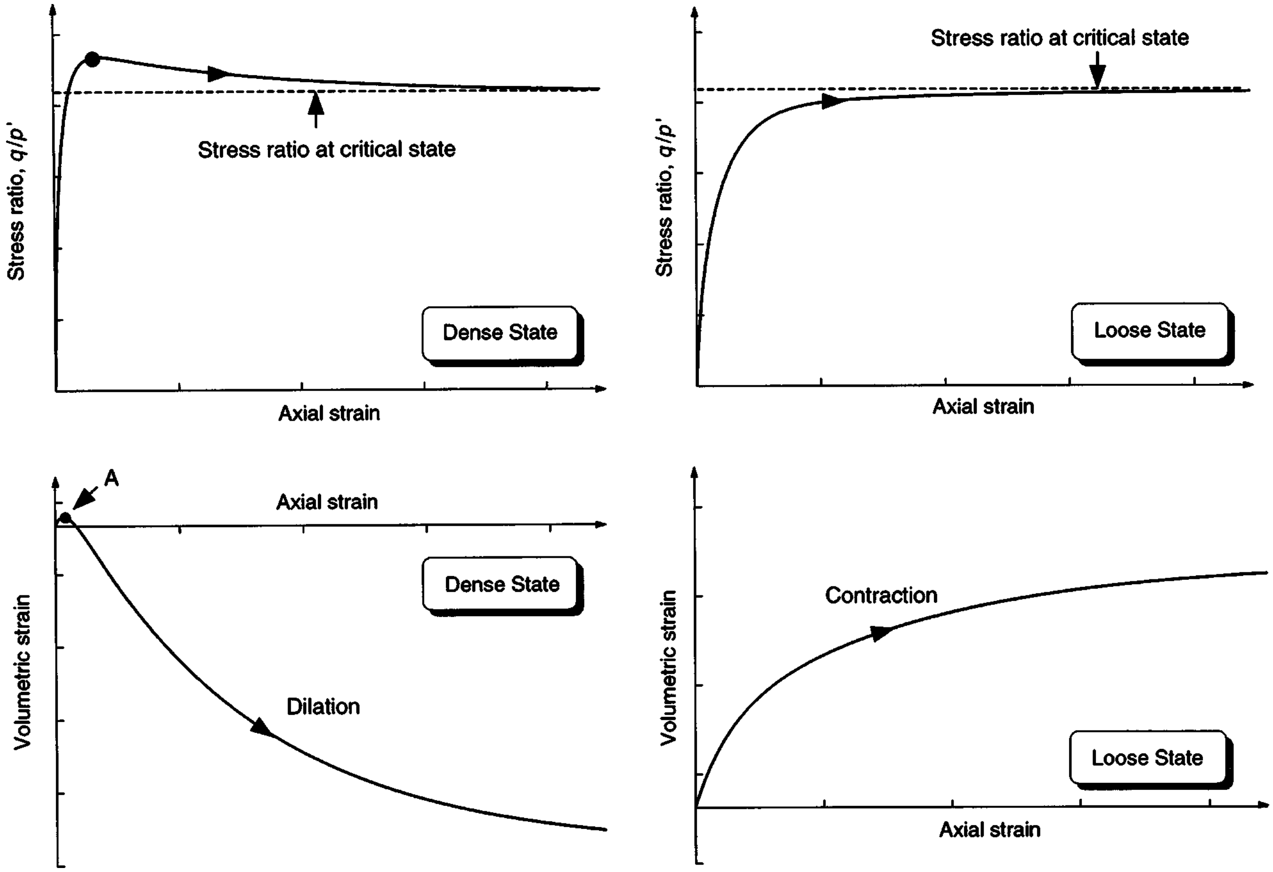

Fig. 1. Typical response of sand in loose and dense states in triaxial compression

ticles. Rowe's stress–dilatancy relation has been widely adopted in many later studies and commonly employed as a flow rule in constitutive modeling of sand behavior (e.g., Nova and Wood 1979; Pastor et al. 1990; Jefferies 1993; Wood et al. 1994). However, the classical stress–dilatancy relation suggested by Rowe (1962) does not allow capturing the important feature of density and pressure dependence because it ignores the dependence of dilatancy on the material internal state. This simplification leads to the common practice in constitutive modeling that treats a *sand* with different initial densities as *different* materials and results in multiple sets of parameters for a single sand. This treatment apparently does not have a good control over changes in the material state during the shearing.

In recent years attempts have been made to tackle this key issue in sand modeling (Gudehus 1996; Manzari and Dafalias 1997; Gajo and Wood 1999; Wan and Guo 1999; Li and Dafalias 2000; Li 2002). As pointed out recently by Li and Dafalias (2000), a dilatancy function without a material state dependence is a fundamental obstacle to unified modeling of the behavior of granular soils over a wide range of densities and stress levels. Within the framework of critical state soil mechanics, they presented a simple model in triaxial stress space by incorporating the state dependence of dilatancy, whose simulative capability was shown by matching a suite of triaxial test data on Toyoura sand (Verdugo and Ishihara 1996) over a wide range of densities and confining pressures.

The objectives of this paper are, in the framework of unified modeling, (1) to clarify how the peak strength, critical state strength, dilation, and internal state of sand are linked during the deformation history; (2) to explore whether a unique relationship could be established between the peak friction angle and dilation angle for a variety of combinations of densities and stress levels through the state-dependent dilatancy; and (3) to identify the deficiency of classical stress–dilatancy relations such as Rowe's and Cam-clay's relations in modeling the state-dependent sand behavior. This study can be regarded as a necessary step towards a full understanding of the density and pressure dependent strength of sands, which so far has been discussed largely from the experimental and empirical points of view (Bolton 1986; Vaid and Sasitharan 1992).

State-Dependent Dilatancy and Unified Modeling

Drawbacks of Rowe's Stress–Dilatancy Relation

Considering a triaxial sample that is being sheared under a set of major and minor principle stresses σ_1 and σ_3 as shown in Fig. 2, Rowe (1962) suggested an expression that states that the ratio of the work done by the driving stress to the work done by the driven stress in any strain increment should be constant, that is

$$\frac{E_{\text{in}}}{E_{\text{out}}}=\frac{\sigma_1' d\varepsilon_1}{-2\sigma_3' d\varepsilon_3}=-\bar{K} \tag{2}$$

where $d\varepsilon_1$ and $d\varepsilon_3$ = strain increments in the axial and radial directions, respectively; and the constant $\bar{K}$ is related to an angle of friction $\bar{\phi}_f$ as follows:

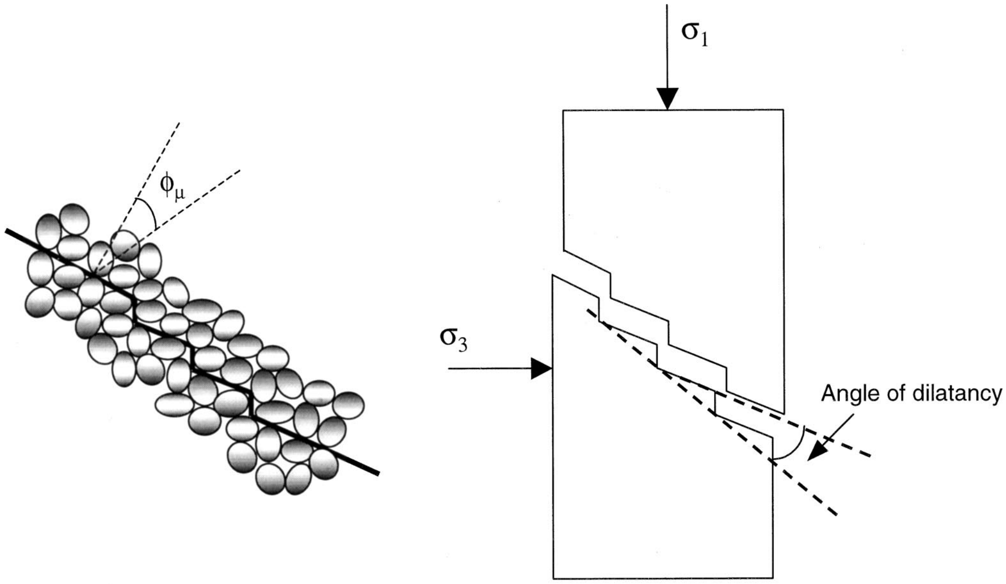

Fig. 2. Sliding mechanism assumed for Rowe's stress–dilatancy theory

$$\overline{K}=\tan^2\left(\frac{\pi}{4}+\frac{\overline{\phi_f}}{2}\right) \tag{3}$$

The angle $\overline{\phi_f}$ represents an equivalent friction angle that varies between the intrinsic interparticle friction angle ϕ_μ and the macroscopic critical state angle of friction for constant volume shearing ϕ'_{cs} such that $\phi_\mu \leqslant \overline{\phi_f} \leqslant \phi'_{cs}$.

Introducing the stress variables and strain increments commonly defined in a triaxial setting, Eq. (2) can be rewritten as

$$\frac{d\varepsilon_v}{d\varepsilon_q}=\frac{3\eta(2+\overline{K})-9(\overline{K}-1)}{2\eta(\overline{K}-1)-3(2\overline{K}+1)} \tag{4}$$

Here $d\varepsilon_v$ = volumetric strain increment; $d\varepsilon_q$ = deviator strain increment; and $\eta=q/p'$ = stress ratio of the deviator stress q to the mean effective stress p'.

In general, the equivalent friction angle $\overline{\phi_f}$ is often taken as the friction angle at critical state ϕ'_{cs} and the elastic strains are much smaller than the plastic strains. Hence, Eq. (4) can further be given in an alternative form as

$$\frac{d\varepsilon_v^p}{d\varepsilon_q^p}=\frac{9(M-\eta)}{9+3M-2M\eta} \tag{5}$$

where M is the critical stress ratio that is related to the critical state angle by

$$\sin\phi'_{cs}=\frac{3M}{6+M} \tag{6}$$

Apparently, Eq. (5) has some similarity to the original Cam-clay flow rule that is well known as

$$\frac{d\varepsilon_v^p}{d\varepsilon_q^p}=M-\eta \tag{7}$$

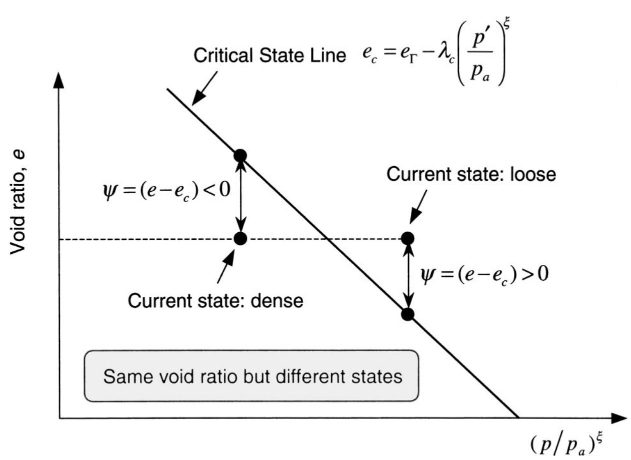

Fig. 3. State parameter and critical state line

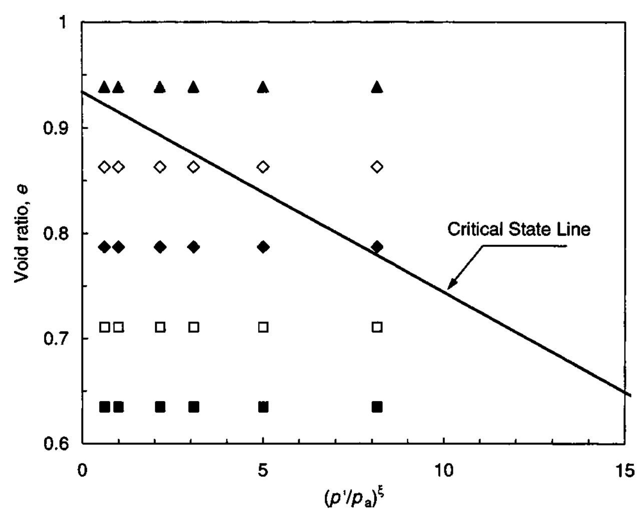

Fig. 4. Initial states of Toyoura sand covered in analyses

The similarity lies in that both the Rowe's equation and the Cam-Clay's equation define a unique relationship between the stress ratio η and the dilatancy $d=d\varepsilon_v^p/d\varepsilon_q^p$. Mathematically, this relationship can be written in a general form as

$$d=f(\eta,C) \tag{8}$$

where C=set of intrinsic material constants. The function expressed by Eq. (8) implies that the soil yielding at $\eta=M$ is coincident with $d=0$; that is, the material being modeled reaches its ultimate failure whenever a plastic deformation takes place at $\eta=M$. However, this stress–dilatancy relation is not always in agreement with the experimental observations. Taking the typical drained behavior shown in Fig. 1 as an example, for the sand in dense state subjected to shear loading, the dilatancy may become zero before the sand reaches its critical state, that is, $d=0$ but $\eta\neq M$. In fact, ignorance of the dependence of dilatancy on the internal state of the material in the classical stress-dilatancy relations is the major obstacle to unified modeling of sand behavior.

State-Dependent Dilatancy

Based on the observations on a number of features in sand shear response and a simple micromechanical analysis, Li and Dafalias (2000) proposed a general expression of the state-dependent dilatancy

$$d=f(\eta,e,Q,C) \tag{9}$$

in which Q represents internal state variables other than the void ratio e and intrinsic material constants C. Eq. (9) expresses the dependence of d on the state variables, which consist of the external variable η and the internal variables e and Q. This state-dependent dilatancy function provides a general framework of a flow rule in plasticity.

To formulate a specific dilatancy function certain requirements are to be satisfied. First, the dilatancy must be zero at critical state, that is

$$d=f(\eta=M,e=e_c,Q,C)=0 \tag{10}$$

where e_c=void ratio at critical state. It is to be noted that Eq. (10) implies the condition $\eta=M$ alone does not guarantee that the critical state has been reached, which however is assumed in the classical stress-dilatancy relations as discussed earlier.

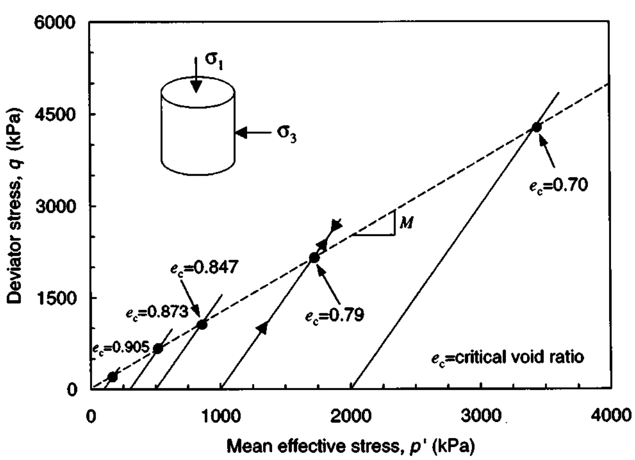

Fig. 5. Typical stress paths involved in analyses

Table 1. Physical Properties of Toyoura Sand

Property	Value
Mean grain size, D_{50} (mm)	0.17
Uniformity coefficient, U_c	1.7
Maximum void ratio, $e_{\max}$	0.977
Minimum void ratio, $e_{\min}$	0.597
Specific gravity, G_s	2.65
Fines content	0%

Second, as pointed out before, the dilatancy function must allow a characteristic state at which $d=0$ but $\eta\neq M$ and $e\neq e_c$. Mathematically, this state can be expressed as

$$d=f(\eta\neq M,e\neq e_c,Q,C)=0 \tag{11}$$

Within the framework of the general expression Eq. (9) subjected to the requirements in Eqs. (10) and (11), the internal variables Q are to be quantified to obtain the dilatancy. As discussed earlier, the relative density on its own is not sufficient to describe the state of a sand, on which the dilatancy depends. Both density and stress level should rather be taken into account. The state parameter proposed by Been and Jefferies (1985), which is defined as $\psi=e-e_c$, the difference between the current void ratio and the critical void ratio corresponding to the current mean effective stress (see Fig. 3), is employed to describe the state of a sand. Note that the critical state line is defined as the following in order to improve the fitting with experimental data for sands

$$e_c=e_\Gamma-\lambda_c\left(\frac{p'}{p_a}\right)^{\xi} \tag{12}$$

where e_Γ, λ_c, and ξ=material constants determining the critical state line in the $e-p'$ plane. Apparently, ψ is a measure of how far the current state is from the critical state. If ψ is negative, the sand is considered in a dense state, and on the contrary, if ψ is positive, the sand is in a loose state. This state description makes it possible that even two samples of a sand with the same void ratio may stay in different sates, as clearly illustrated in Fig. 3. A particular form of dilatancy function that incorporating the state dependence can be suggested as follows (Li and Dafalias 2000):

$$d=d_0\left(\exp(m\psi)-\frac{\eta}{M}\right) \tag{13}$$

in which d_0 and m=two material constants.

Unified Modeling in Triaxial Stress Space

It is assumed that the strain components in a triaxial setting can be written as

Table 2. Model Parameters Calibrated for Toyoura Sand

Elastic parameters	Critical state parameters	Dilatancy parameters	Hardening parameters
$G_0=125$	$M=1.25$	$d_0=0.88$	$h_1=3.15$
	$e_\Gamma=0.934$	$m=3.5$	$h_2=3.05$
$\nu=0.05$	$\lambda_c=0.019$		$n=1.1$
	$\xi=0.7$		

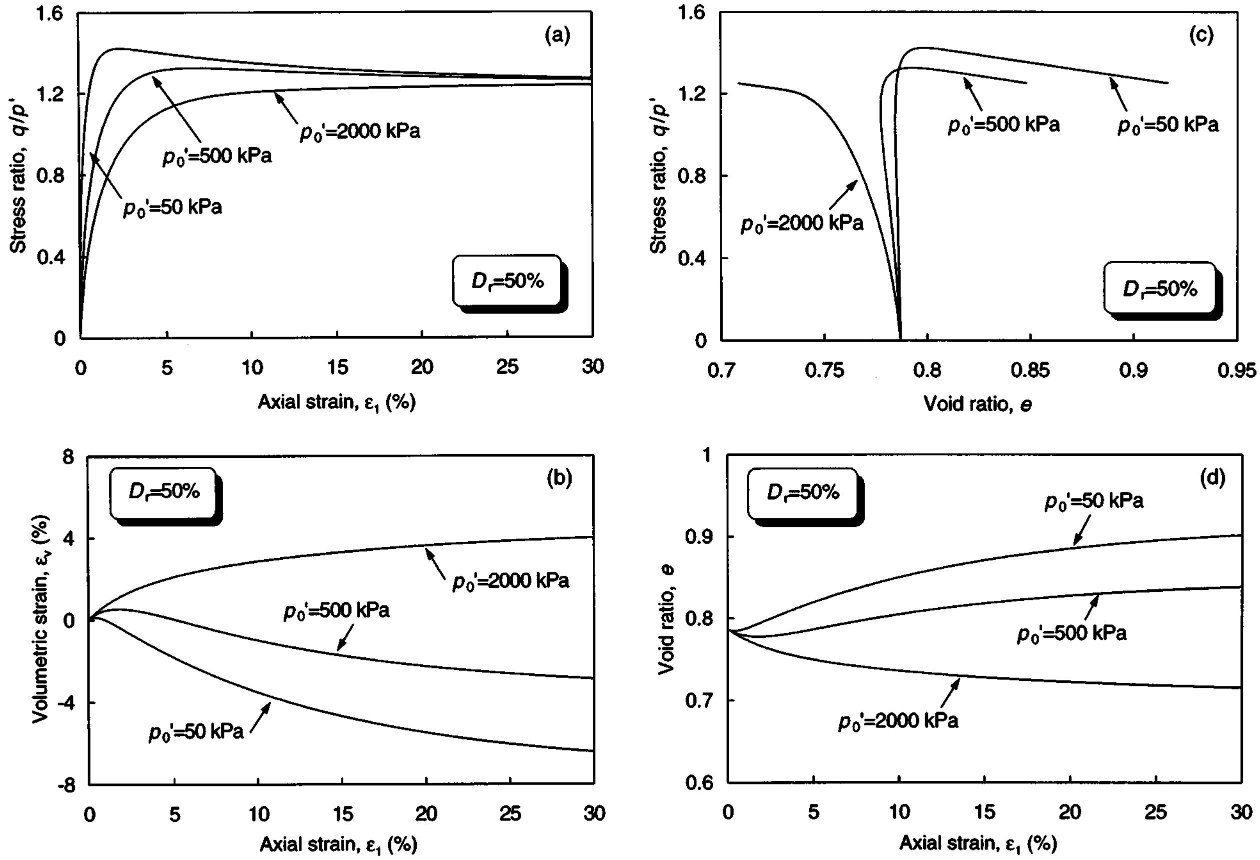

Fig. 6. Typical stress–strain response and volume change characteristics of sand with different confining pressures

$$d\varepsilon_q = d\varepsilon_q^e + d\varepsilon_q^p \tag{14}$$

$$d\varepsilon_v = d\varepsilon_v^e + d\varepsilon_v^p \tag{15}$$

where superscripts "e" and "p" stand for elastic and plastic, respectively.

Introducing the yield criterion as

$$f = q - \eta p' = 0 \tag{16}$$

and with the definition of a loading index (Li and Dafalias 2000), the plastic strain increments can be given as

$$\begin{Bmatrix} d\varepsilon_q^p \\ d\varepsilon_v^p \end{Bmatrix} = \begin{Bmatrix} \dfrac{p' d\eta}{K_p} \\ \dfrac{p' d\eta}{K_p} d \end{Bmatrix} \tag{17}$$

where K_p = plastic hardening modulus and $d = d\varepsilon_v^p / d\varepsilon_q^p$. Note that $d > 0$ indicates volumetric contraction while $d < 0$ indicates dilation.

Eq. (17) implies that a constant η path induces no plastic deformation. This is, of course, only approximately true, but it is still a good approximation in many cases since under normal levels of confining pressures of interest, a constant η path only induces a relatively small plastic volume change in sands, before grain-crushing levels of pressures are reached as corroborated experimentally by Poorooshasb et al. (1966, 1967). For a fully fledged model where the plastic deformations under constant η are to be considered, additional mechanisms, such as a p' controlling cap, can be added (Li 2002).

With Hooke's law, the elastic strain increments can be determined and, finally, the relationship between stress and strain increments can be established as

$$\begin{Bmatrix} dq \\ dp' \end{Bmatrix} = \left[\begin{pmatrix} 3G & 0 \\ 0 & K \end{pmatrix} - \frac{h(L)}{3G - K\eta d + K_p} \begin{pmatrix} 9G^2 & -3KG\eta \\ 3KGd & -K^2\eta d \end{pmatrix} \right] \times \begin{Bmatrix} d\varepsilon_q \\ d\varepsilon_v \end{Bmatrix} \tag{18}$$

where $h(L)$ = Heaviside function; the elastic moduli G and K, and the plastic modulus K_p are given, respectively, as

$$G = G_0 \cdot \frac{(2.973 - e)^2}{1 + e} \cdot \sqrt{p' p_a} \tag{19}$$

$$K = G \cdot \frac{2(1+\nu)}{3(1-2\nu)} \tag{20}$$

$$K_p = G(h_1 - h_2 e)\left(\frac{M}{\eta} - \exp(n\psi)\right) \tag{21}$$

in which G_0 = material constant; ν = Poisson's ratio; p_a = atmospheric pressure; and h_1, h_2, and n = model parameters. It is to be noted that the dependence on the state parameter has been introduced into the plastic modulus.

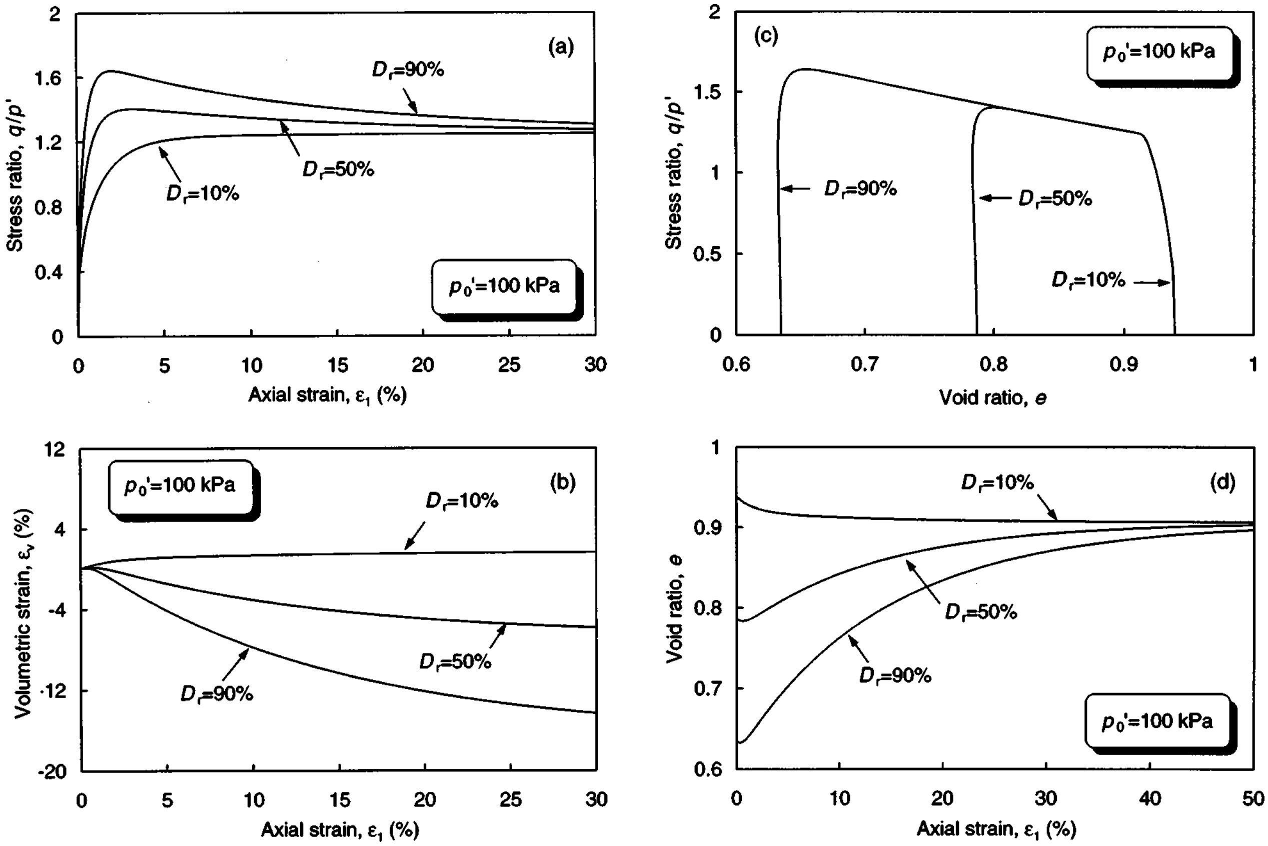

Fig. 7. Typical stress–strain response and volume change characteristics of sand with different initial densities

State-Dependent Shear Strength of Sands

The procedure for calibration of model constants and the simulative capability of the model have been shown for Toyoura sand (Li and Dafalias 2000), for which well-documented triaxial test results were available (Verdugo and Ishihara 1996). Here, efforts are made to explore how the peak strength, critical state strength, and dilation are linked during the deformation history and whether a unique relationship exists between the peak friction angle and dilation angle for a variety of combinations of density and confining pressure. Within the scope of the present paper, all discussions are restricted to the drained behavior.

Five typical values of initial void ratio/density of Toyoura sand, i.e., $D_r = 10$, 30, 50, 70, and 90%, and six values of confining pressure, i.e., $p_0' = 50$, 100, 300, 500, 1,000, and 2,000 kPa are combined such that a wide range of loose and dense states are covered, as shown in Fig. 4 where the critical state line is also presented in the $e-p'$ plane. The typical stress paths in $q-p'$ plane are illustrated in Fig. 5. The physical properties of the Toyoura sand are described in Table 1 and the model parameters that have been carefully calibrated for this sand are given in Table 2.

Stress–Strain–Volume Behavior as Affected by Density and Pressure

Fig. 6 illustrates the typical stress ratio–axial strain and volume change behavior of the sand with an initial density $D_r = 50\%$ ($e = 0.787$) but subjected to different mean effective stresses, ranging from as low as 50 kPa to as high as 2,000 kPa. The high stress level may represent the field conditions such as that under large dams. It is clear that while the sand is at the same density, it may exhibit quite different response for different confining pressures. When the sand is confined at a low pressure of 50 kPa, a peak stress ratio and strain softening from this peak can be observed; the sand contracts firstly but soon begins to dilate [see Fig. 6(b)]. For the sand initially confined at a very high pressure, 2,000 kPa, a strain hardening response appears without a peak stress ratio developed. For all these three cases of confining pressure, the stress ratio q/p' can eventually reach the same critical stress ratio but with different critical void ratios as shown in Figs. 6(c and d).

The influence of initial density on the behavior of the sand is shown in Fig. 7 for a confining pressure of 100 kPa. For the sand initially in dense states, i.e., $D_r = 90\%$ ($e = 0.635$) and $D_r = 50\%$ ($e = 0.787$), a peak stress ratio appears at the early stage of deformation, followed by a strain softening [Fig. 7(a)]; correspondingly the volume change is found to be contractant first and dilatant subsequently. For the sand initially in a loose state ($D_r = 10\%$ or $e = 0.939$), however, no peak appears and a strain hardening is observed during the deformation history. Since all the three samples are sheared at the same confining pressure, an identical critical void ratio is reached when the stress ratio q/p' approaches the critical stress ratio at large strains as shown in Figs. 7(c and d).

It should be noted that all the behavior observed for the variety of combinations of density and pressure are produced with a single set of model constants, which differentiates from the common constitutive models that treat a *sand* with the same intrinsic properties but different initial states as *different* materials and thereby represents a rational unified modeling.

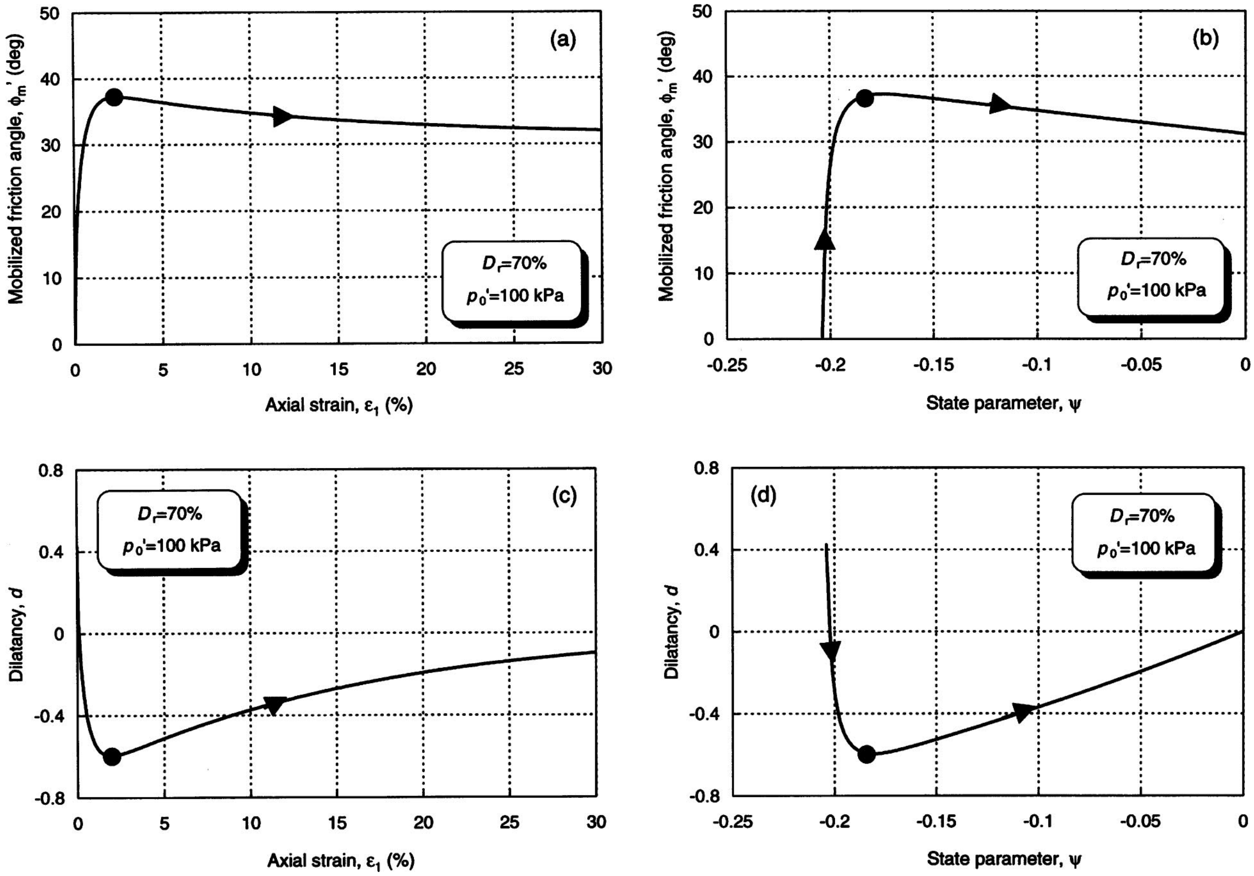

Fig. 8. Evolution of mobilized friction angle and dilatancy for dense state

Evolution of Mobilized Angle and Dilatancy during Deformation

Figs. 8 and 9 illustrate the evolution of the mobilized friction angle and the dilatancy during shearing for two typical cases respectively: one is that the sand in a dense state ($D_r = 70\%$ and $p_0' = 100$ kPa) and the other is the sand in a loose state ($D_r = 10\%$ and $p_0' = 1{,}000$ kPa). The mobilized angle of shearing resistance is related to the stress ratio $\eta = q/p'$ by

$$\sin \phi_m' = \frac{3\eta}{(6+\eta)} \tag{22}$$

For the sand in a dense state, it is evident from Fig. 8(a) that the angle of friction is mobilized very fast at the early stage of deformation, with a peak friction angle of about 38° developed. This peak angle is followed by a reduction in the shearing resistance eventually to the critical state angle of 31.15° at large levels of strain.

More interestingly, Fig. 8(b) shows the mobilization of the friction angle with the state parameter ψ during the shear. At the beginning of the deformation, the sand is in a dense state characterized by a negative state parameter $\psi_0 = -0.204$. It can be seen that, essentially, before the peak angle is mobilized there is no change in the state parameter as the sand is sheared. However, immediately following the development of the peak angle, a significant change in the state parameter takes place, with a reduction in the magnitude of the state parameter ending at the critical state at which the state parameter is zero.

The evolution of dilatancy shown in Figs. 8(c and d) sheds lights on the mechanism of the state-dependent strength. At the beginning of shearing, the dilatancy d takes a positive value; this is reasonable because the sand is in a dense state with a negative initial state parameter ψ_0. This positive dilatancy makes sure the contractive response takes place at the beginning. As shearing goes on, the dilatancy evolves from positive values to negative values; it becomes zero for the first time at the characteristic state that is associated with a maximum compression. It is clear from Fig. 8(d) that this state is far from the critical state where $\psi = 0$ and $\eta = M$. A maximum dilatancy can develop quickly after the characteristic state is passed; following this maximum value the magnitude of dilatancy gradually reduces as shear proceeds and it finally becomes zero at the critical state.

The mobilization of friction angle and the evolution of dilatancy during the deformation for the sand in loose states are quite different from that described above, as can be seen from Fig. 9. The magnitude of dilatancy decreases as the sand is sheared and it becomes zero at the final stage that is associated with large strain levels. d takes positive values throughout the deformation, implying that volumetric contraction occurs as shearing proceeds. The friction angle is mobilized increasingly as the shear strain develops, with the critical state angle as its upper limit.

Evolution of Stress Ratio with Dilatancy

The evolution of the stress ratio $\eta = q/p'$ with dilatancy is presented in Fig. 10(a) for the sand with $D_r = 50\%$ sheared under different mean effective stresses. Fig. 10(b) shows the stress

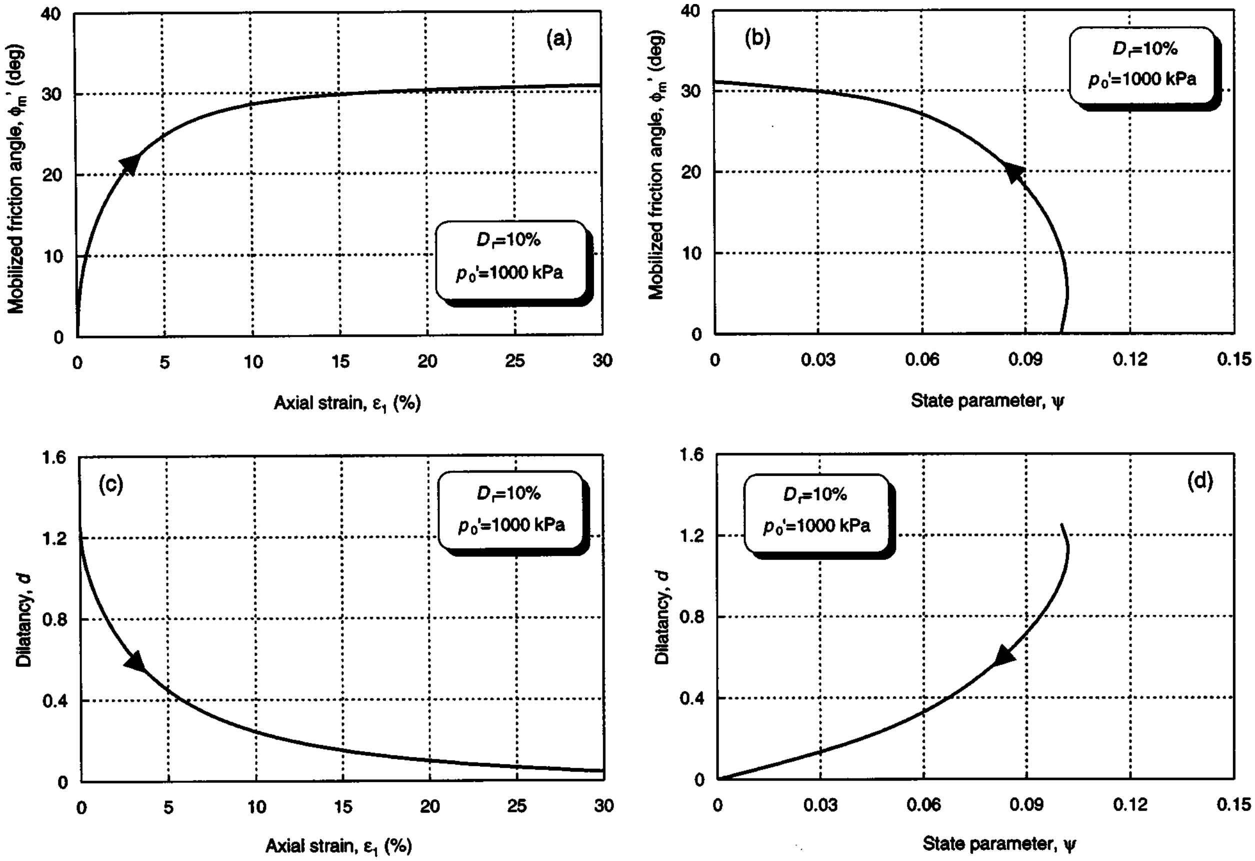

Fig. 9. Evolution of mobilized friction angle and dilatancy for loose state

ratio–dilatancy curves for the sand with different relative densities but subjected to the same confining pressure, 100 kPa. The plots suggest that there is no unique relationship between the stress ratio and the dilatancy, but rather a family of curves exist for different densities and stress levels. For the sand in dense/dilative states, the stress ratio–dilatancy curves display a bend which corresponds to the peak stress ratio and maximum dilation, and furthermore the peak stress ratio/maximum dilatancy depends on the initial density and confining pressure.

For the sand with a specified density, the lower the confining pressure the greater the maximum dilatancy. On the other hand, for the sand confined at the same pressure but with different densities, the larger the relative density the greater the maximum dilatancy. As will be shown later, the influence of density and stress level can be combined through the state parameter. As far as the sand initially at loose states is concerned, it can be seen that volumetric expanding does not occur throughout the shearing process, i.e., d is positive. All the curves for both dense/dilative and loose/contractive states in the stress–dilatancy plots are found to converge at the critical state where $\eta = M$ and $d = 0$. It is worth noting that the theoretical prediction of the stress–dilatancy behavior described above agrees very well with the experimental observations on the Toyoura sand as shown in Fig. 11. The data points are generated based on the test data by Verdugo and Ishihara (1996).

To clearly identify the deficiency of the classical stress-dilatancy relations, Fig. 12 shows several typical stress–dilatancy curves together with those obtained using Rowe's and Camclay's relations as presented earlier. It is evident that both relations cannot account for the influence of density and stress level although the Rowe's relation performs better than the Cam-clay's relation. The prediction by Rowe's relation seems to only represent the stress–dilatancy behavior of the sand in loose states.

State-Dependent Peak Friction Angle and Dilation Angle

Fig. 13 shows the influence of initial density/void ratio and confining pressure on the peak friction angle, ϕ_p'. It is clear that the peak angle of friction decreases steadily with increasing initial void ratio [see Fig. 13(a)]; at a specified void ratio, the peak friction angle is higher for lower confining pressures. A steady decrease in the peak angle of friction as the mean effective stress increases can be observed in Fig. 13(b). At a specified stress level, a larger peak friction angle can be achieved for the sand with a higher density. In both plots the critical state angle of friction provides the low limit. A similar tendency has been observed in laboratory tests on several different sands (Bolton 1986), as shown in Fig. 14. for Berlin sand (De Beer 1965).

The influence of density and confining pressure on the maximum dilation angle is illustrated in Fig. 15. The maximum dilation angle is defined herein as

$$\sin \theta_{\max} = \frac{2}{3}\left(\left|\frac{d\varepsilon_v^p}{d\varepsilon_p^q}\right|\right)_{\max} = \frac{2}{3}(|d|)_{\max} \tag{23}$$

It is interesting to note that a very similar influence of initial density/void ratio and mean effective stress exists on the dilation angle. A steady decrease of the maximum dilation angle occurs as the initial void ratio or the confining pressure increases. This find-

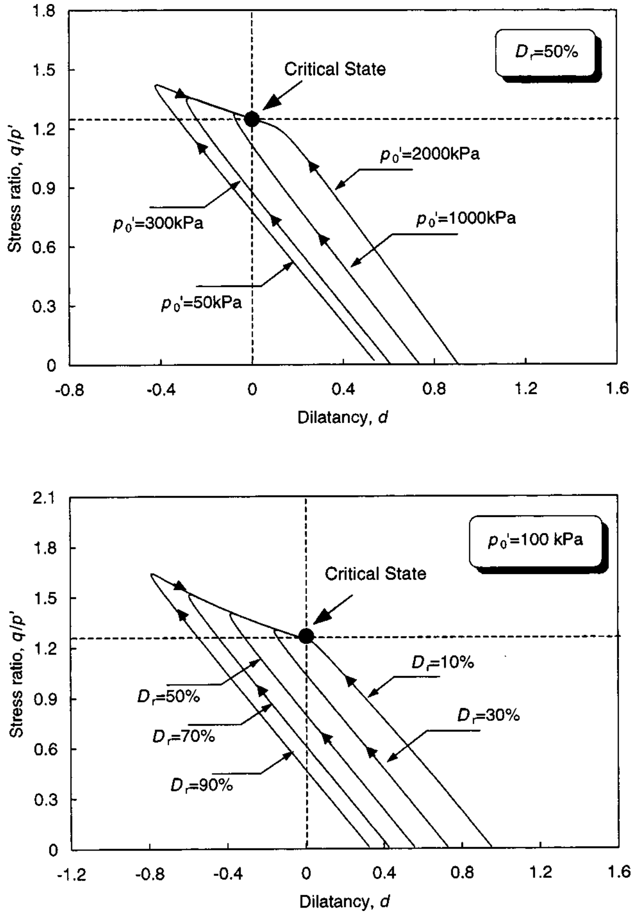

Fig. 10. Development of stress ratio with dilatancy for sand with different confining pressures and densities

ing implies that a unique relationship between the peak friction angle and the maximum dilation angle might exist.

With the aid of Eqs. (13) and (21), the relationships between the peak friction angle, the maximum dilation angle, and the state parameter can be established analytically, as shown by solid lines in Fig. 16. It is to be noted that the state parameters so established are those corresponding to the peak states, ψ_p. In engineering practice, however, it is difficult to directly determine ψ_p accurately. For this reason, the influence of density and stress level on the peak angle of shearing resistance and the maximum dilation angle is combined herein through the initial state parameter, ψ_0, which describes the initial location relative to the critical state line. Approximate relationships represented by the dashed lines are suggested, for the purpose of practical applications, to describe the general trend for the data in Fig. 16: both the peak friction angle and the dilation angle decrease as the magnitude of initial state parameter decreases. The critical state angle for constant volume and the zero initial state parameter provide two bounds in the $\phi_p'-\psi_0$ plot, and the bounds in the $\theta_{\max}-\psi_0$ plot are provided by $\psi_0=0$ and $\theta_{\max}=0$. The theoretical prediction of the relationships among the peak strength, the maximum dilation, and the state parameter is found to be consistent with the overall tendency exhibited by data collected from quite a few triaxial tests on different sands (Been and Jefferies 1986; Been et al. 1992), as shown in Fig. 17, although scatters exist in these data. More experimental data of high quality is desirable in order to improve the interpretation presented here.

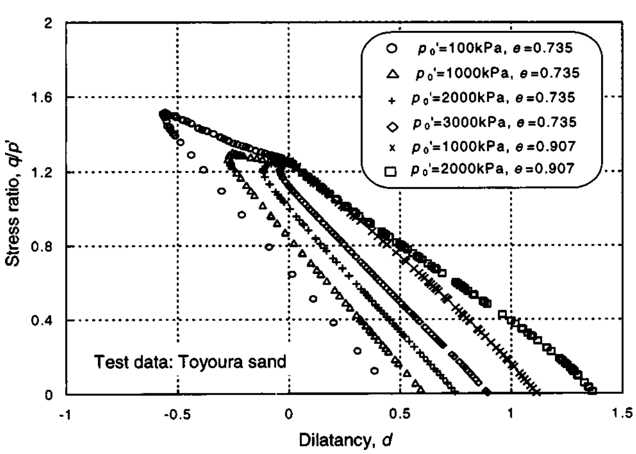

Fig. 11. Experimental data of stress–dilatancy relation (data from Verdugo and Ishihara 1996)

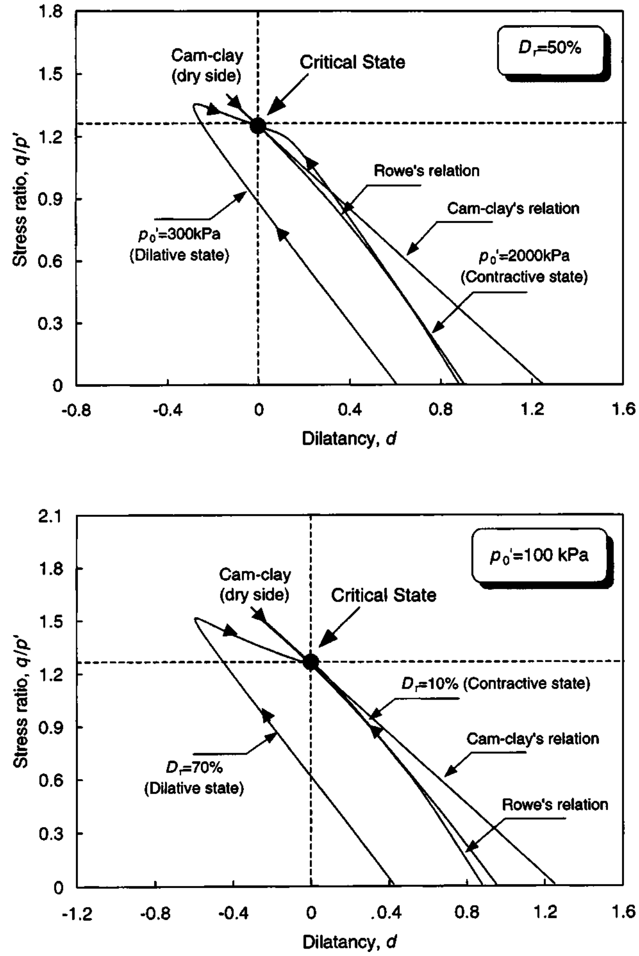

Fig. 12. Deficiency of Rowe's and Cam-clay's relations in stress–dilatancy plots

Fig. 18 shows the relationship between the peak friction angle and the maximum dilation angle obtained for a variety of combinations of initial void ratios and confining pressures. It is evident that a linear relationship may be sufficient to describe the observed trend: the peak friction angle increases with increasing the dilation angle, and it becomes the critical state angle ϕ_{cs}' when the dilation angle is zero. The linear relationship is proposed as

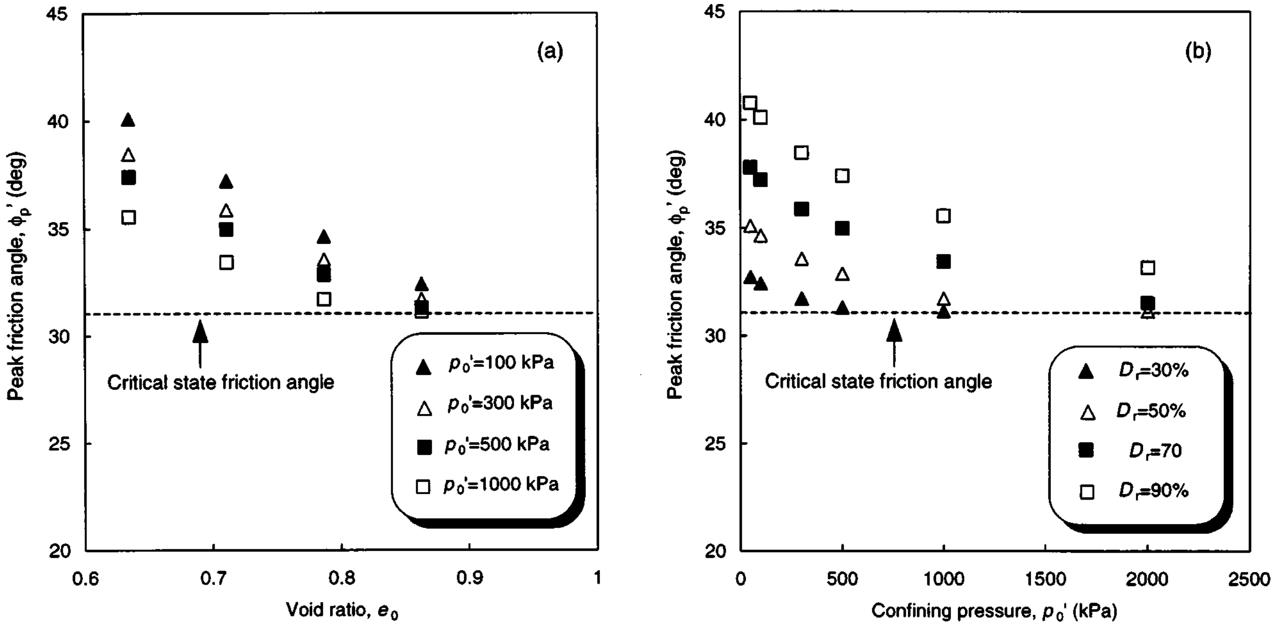

Fig. 13. Influence of density and stress level on peak friction angle

$$\phi_p' = \phi_{cs}' + 0.28\theta_{\max} \tag{24}$$

It is interesting to note that a similar relation was observed recently by Vaid and Sasitharan (1992) from their triaxial tests on Erksak sand (see Fig. 19); in their tests the excess friction angle $(\phi_p' - \phi_{cs}')$ was found to be only one-third of the maximum dilatation angle. Based on data from plane strain compression tests on various types of sand, Bolton (1986) proposed an empirical, linear relation between the peak friction angle and the maximum dilation angle as

$$\phi_p' = \phi_{cs}' + 0.8\theta_{\max} \tag{25}$$

Comparison of Eq. (24) with Eq. (25) implies that the excess friction angle $(\phi_p' - \phi_{cs}')$ in triaxial conditions is about 35% of that in plane strain conditions.

It should be pointed out that although the analyses presented above have provided insightful information on the key aspect of the strength of granular soils, there remain several issues that cannot be overlooked in some applications and hence need to be investigated further. One is the dependence of strength on the shearing mode, that is, the frictional angles at triaxial compression, simple shear, as well as plane strain may be different, with plane strain usually a few degrees higher than that in triaxial compression. The other issue is the fabric effects on the shear strength. Some experimental observations have become available on these interesting features of granular soils (e.g., Yoshimine et al. 1998). Further discussions from the viewpoint of unified constitutive modeling may give a better understanding. To rationally describe these features, provisions such as the dependence on the full stress invariants and appropriate fabric tensors need to be included.

Summary and Conclusions

The density and pressure dependent shear strength of sands has been discussed from the perspective of a unified modeling in triaxial stress space. The unified modeling accounts for the depen-

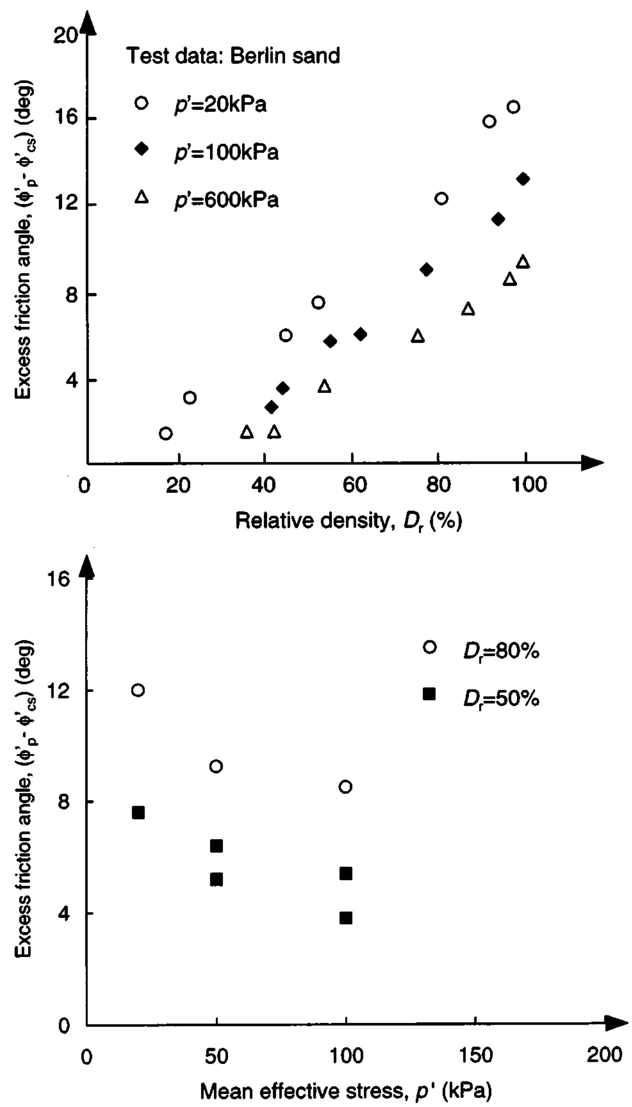

Fig. 14. Test data for density and pressure dependent friction angle (data from De Beer 1965)

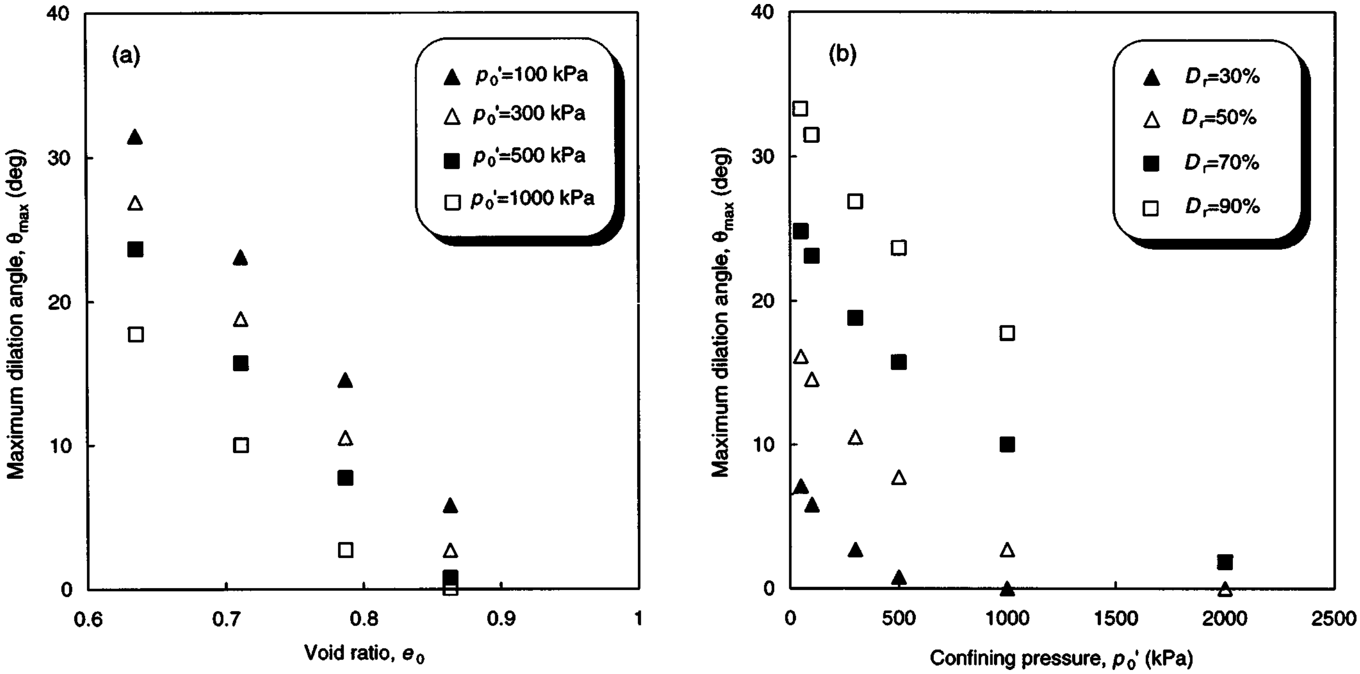

Fig. 15. Influence of density and stress level on maximum dilation angle

dence of dilatancy on the material internal state during the deformation history and thus has the capability of capturing the essential behavior of a sand with different initial densities and confining pressures using a single set of model constants. Based on the analyses the major conclusions can be drawn as follows.

1. For a sand initially in dense states, the friction angle can be mobilized very fast to its peak value at the early stage of deformation while the dilatancy d evolves from positive values (contraction) to negative values (dilation). The dilatancy becomes zero for the first time at the characteristic state that is associated with a maximum compression but is far from the critical state where $\psi=0$ and $\eta=M$. A maximum dilation takes place at this early stage, followed by a gradual reduction in the magnitude of the dilatancy and correspondingly a reduction in shearing resistance as shearing proceeds.
2. A stable decrease of the peak friction angle and the dilation angle occurs as the initial void ratio or confining pressure increases, and the influence of density and stress level can be combined through the initial state parameter ψ_0, a quantity describing the initial location relative to the critical state line

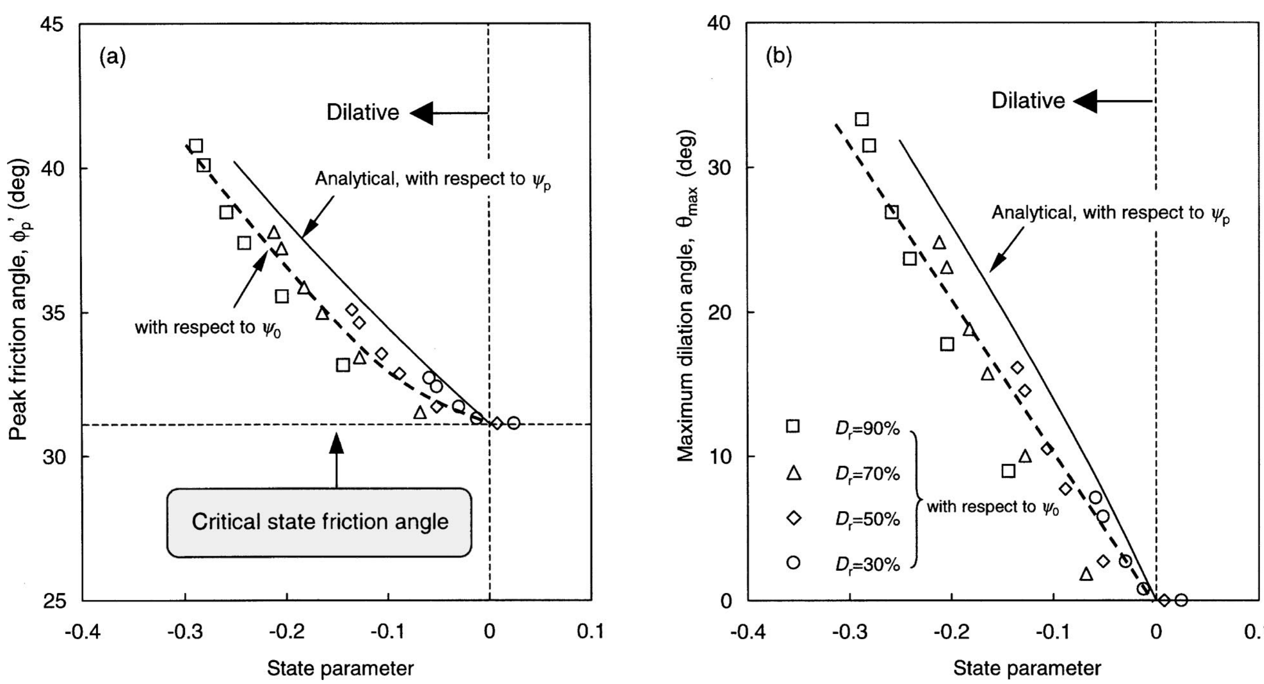

Fig. 16. Peak friction angle and maximum dilation angle as function of state parameter

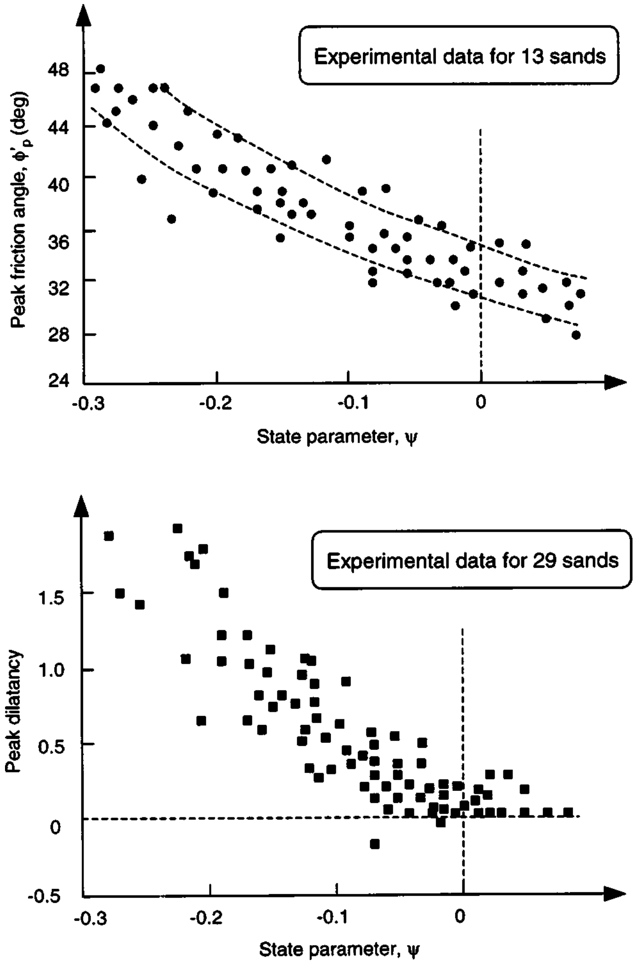

Fig. 17. Experimental data of relations among peak friction angle, maximum dilation, and state parameter (after Been and Jefferies 1986 and Been et al. 1992)

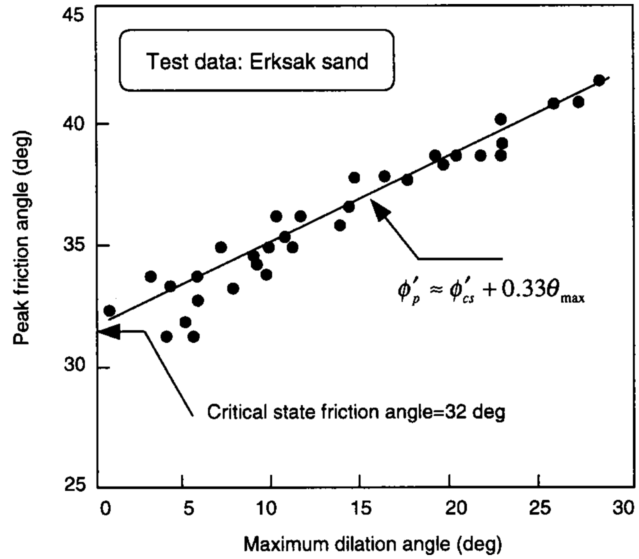

Fig. 19. Test data of relationship between peak friction angle and maximum dilation angle (after Vaid and Sasitharan 1992)

in the $e-p'$ plane. In general, both the peak friction angle and the dilation angle are found to decrease as the magnitude of initial state parameter decreases, showing reasonable agreement with experimental observations.

3. A unique, linear relationship has been suggested between the excess friction angle $(\phi'_p - \phi'_{cs})$ and the maximum dilation angle, which implies that the excess friction angle in triaxial conditions is less than 40% of that in plane strain conditions. This theoretical prediction is found to be consistent with recent experimental findings.
4. The unique relationship between the stress ratio η and dilatancy *d*, as assumed by Rowe's and Cam-clay's stress–dilatancy relations and widely followed in many sand mod-

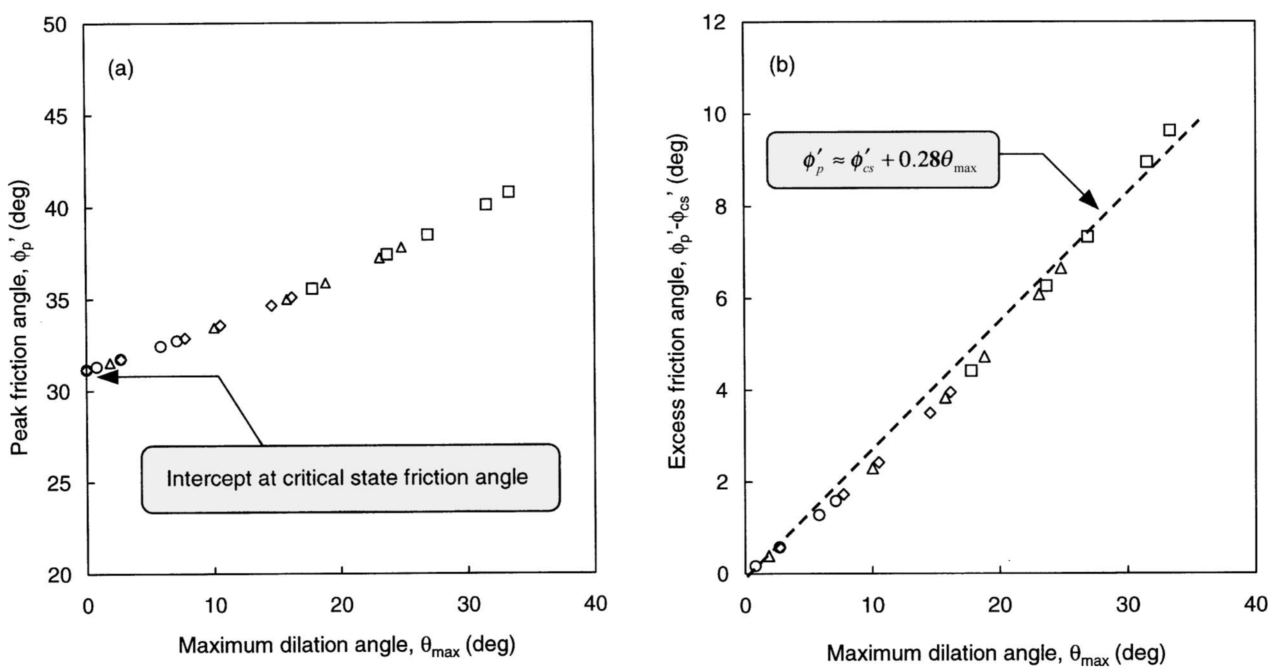

Fig. 18. Relationship between peak friction angle and maximum dilation angle

els, does not exist and thereby obstructs unified modeling of the sand behavior over a wide range of densities and stress levels.

Notations

The following symbols are used in this paper:

D_r = relative density;
d = dilatancy, $d = d\varepsilon_v^p / d\varepsilon_q^p$;
d_0 = model parameter;
$d\varepsilon_q$ = deviator strain increment, $d\varepsilon_q = 2/3(d\varepsilon_1 - d\varepsilon_3)$;
$d\varepsilon_v$ = volumetric strain increment, $d\varepsilon_v = d\varepsilon_1 + 2d\varepsilon_3$;
$d\varepsilon_1, d\varepsilon_3$ = strain increments in axial and radial directions;
$E_{\mathrm{in}}, E_{\mathrm{out}}$ = work done by driving stress and work done by driven stress;
e = void ratio;
e_c = critical void ratio;
e_Γ = critical void ratio intercept at $p' = p_a$;
G = shear modulus;
G_0 = material constant;
h_1, h_2 = model parameters;
$h(L)$ = Heaviside function;
$\overline{K}$ = constant related to angle of friction;
K = bulk modulus;
K_p = plastic modulus;
L = loading index;
M = critical stress ratio;
n = model parameter;
p' = mean effective stress, $p' = (\sigma_1' + 2\sigma_3')/3$;
p_a = atmospheric pressure;
q = deviator stress, $q = \sigma_1 - \sigma_3$;
η = stress ratio, $\eta = q/p'$;
$\theta_{\max}$ = maximum dilation angle;
λ_c = slope of critical state line;
ν = Poisson's ratio;
ξ = material constant;
σ_1, σ_3 = major and minor principal stresses;
ϕ_{cs}' = critical state angle of friction;
$\overline{\phi}_f$ = equivalent friction angle;
ϕ_m' = mobilized angle of friction;
ϕ_p' = peak friction angle;
ϕ_μ = intrinsic interparticle friction;
ψ = state parameter, $\psi = e - e_c$; and
ψ_0 = initial state parameter.

References

Been, K., and Jefferies, M. G. (1985). "A state parameter for sands." *Geotechnique*, 35(2), 99–112.

Been, K., and Jefferies, M. G. (1986). "A state parameter for sands: reply to discussion." *Geotechnique*, 36(1), 123–132.

Been, K., Jefferies, M. G., and Hachey, J. E. (1992). "The critical state of sands: reply to discussion." *Geotechnique*, 42(4), 655–663.

Bishop, A. W. (1971). "Shear strength parameters for undisturbed and remolded soil specimens." *Proc., Roscoe Memorial Symp.*, Cambridge Mass., 3–58.

Bolton, M. D. (1986). "The strength and dilatancy of sands." *Geotechnique*, 36(1), 65–78.

Cornforth, D. H. (1964). "Some experiments on the influence of strain conditions on the strength of sand." *Geotechnique*, 14, 143–167.

Cornforth, D. H. (1973). "Prediction of drained strength of sand from relative density measurements." *Evaluation of relative density and its role in geotechnical projects involving cohesionless soils*, American Society for Testing and Materials, Philadelphia Special Technical Publication 523, 281–303.

De Beer, E. E. (1965). "Influence of the mean normal stress on the shearing strength of sand." *Proc., 6th Int. Conf. Soil Mechanics Foundation Engineering*, Montreal, Vol. 1, 165–169.

Gajo, A., and Wood, D. M. (1999). "Severn-Trent sand: a kinematic-hardening constitutive model: the q-p formulation." *Geotechnique*, 49(5), 595–614.

Gudehus, G. (1996). "A comprehensive constitute equation for granular materials." *Soils Found.*, 36(1), 1–12.

Jefferies, M. G. (1993). "Nor-sand: a simple critical state model for sand." *Geotechnique*, 43(1), 91–103.

Lee, K. L., and Seed, H. B. (1967). "Drained strength characteristics of sands." *J. Soil Mech. Found. Div., ASCE*, 93, 117–141.

Li, X. S. (2002). "A sand model with state-dependent dilatancy." *Geotechnique*, 52(3), 173–186.

Li, X. S., and Dafalias, Y. F. (2000). "Dilatancy for cohesionless soils." *Geotechnique*, 50(4), 449–460.

Manzari, M. T., and Dafalias, Y. F. (1997). "A critical state two-surface plasticity model for sands." *Geotechnique*, 47(2), 255–272.

Nova, R., and Wood, D. M. (1979). "A constitutive model for sand in triaxial compression." *Int. J. Numer. Analyt. Meth. Geomech.*, 3, 255–278.

Pastor, M., Zienkiewicz, O. C., and Chan, H. C. (1990). "Generalized plasticity and the modeling of soil behavior." *Int. J. Numer. Analyt. Meth. Geomech.*, 14, 151–190.

Poorooshasb, H. B., Holubec, I., and Sherbourne, A. N. (1966). "Yielding and flow of sand in triaxial compression, Part I." *Can. Geotech. J.*, 3(5), 179–190.

Poorooshasb, H. B., Holubec, I., and Sherbourne, A. N. (1967). "Yielding and flow of sand in triaxial compression, Part II." *Can. Geotech. J.*, 4(5), 376–397.

Roscoe, K. H., Schofield, A. N., and Wroth, C. P. (1958). "On the yielding of soils." *Geotechnique*, 8(1), 22–53.

Rowe, P. W. (1962). "The stress-dilatancy relation for static equilibrium of an assembly of particles in contact." *Proc. R. Soc. London, Ser. A*, 269, 500–527.

Schofield, A. N., and Wroth, C. P. (1968). *Critical state soil mechanics*, McGraw–Hill, London.

Stroud, M. A. (1971). "*The behavior of sand at low stress levels in the simple shear apparatus*." PhD thesis, Cambridge Univ., Cambridge.

Vaid, Y. P., and Sasitharan, S. (1992). "The strength and dilatancy of sand." *Can. Geotech. J.*, 29, 552–526.

Verdugo, R., and Ishihara, K. (1996). "The steady state of sandy soils." *Soils Found.*, 36(2), 81–91.

Wan, R. G., and Guo, R. G. (1999). "A pressure and density dependent dilatancy model for granular materials." *Soils Found.*, 39(6), 1–12.

Wood, D. M. (1990). *Soil behaviour and critical state soil mechanics*, Cambridge University Press, London.

Wood, D. M., Belkheir, K., and Liu, D. F. (1994). "Strain softening and state parameter for sand modeling." *Geotechnique*, 44(2), 335–339.

Yoshimine, M., Ishihara, K., and Vargas, W. (1998). "Effects of principle stress direction and intermediate principal stress on undrained shear behavior of sand." *Soils Found.*, 38(3), 179–188.